Triangular Self-Assembly

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Abstract

We discuss the self-assembly system of triangular tiles instead of square tiles, in particular right triangular tiles and equilateral triangular tiles. We show that the triangular tile assembly system, either deterministic or non-deterministic, has the same power to the square tile assembly system in computation, which is Turing universal. By providing counter-examples, we show that the triangular tile assembly system and the square tile assembly system are not comparable in general. More precisely, there exists square tile assembly system S such that no triangular tile assembly system is a division of S and produces the same shape; there exists triangular tile assembly system T such that no square tile assembly system produces the same compatible shape with border glues. We also discuss the assembly of triangles by triangular tiles and obtain results similar to the assembly of squares, that is to assemble a triangular of size $O(N^2)$, the minimal number of tiles required is in $O(\log N/\log \log N)$.

1 Introduction

In the nature, molecules tend to interact to form more complicated structures of crystals and supramolecules. The spontaneous construction of particular molecular structures is one important topic in DNA and molecular computing, which is based on the Watson-Crick complementarity between pairs of DNA strands. Generally, the process is composed of two steps. First, the basic building blocks are carefully designed and constructed by synthetic chemistry; and then the aimed large structure is assembled by sticking basic blocks together through Watson-Crick complementarity. In 1996, Winfree [7] showed how the formation of large structures from certain DNA molecules can simulate the Blocked Cellular Automata (BCA), which is of the same computational power to the Turing machines. In 1998, Winfree, Liu, Wenzler, and Seeman [8] designed and produced two-dimensional DNA crystals in their laboratory by the method of self-assembly.

One systematic study on this topic is the self-assembly of squares. In 1999, Adleman [2] proposed models of self-assembly, which are based on the theory of Wang tiles [6], and studied the time complexity of linear polymerization via "step counting". In 2000, Rothemund and Winfree [5] showed that to deterministically self-assemble an $N \times N$ full square, N^2 tiles is required for temperature $\tau = 1$ and the number of tiles for the case of fixed temperature $\tau \geq 2$ is $O(\log N)$. In 2001, Adleman, Cheng, Goel, and Huang [1] showed that $\Theta(\log N/\log \log N)$ tiles is enough for fixed temperature $\tau \geq 2$. In 2006, Kao and Schweller [3] showed that if the temperature τ is allowed to change systematically, then a constant number of tiles is enough for the self-assembly of arbitrary $N \times N$ full square with temperature sequence of length $O(\log N)$.

One variation on the self-assembly of squares is that we study tile of shapes other than squares that can tile a full two-dimensional plane; and instead of considering full squares, we discuss the self-assembly of other particular full two-dimensional region. For tiling a full two-dimensional plane with one single shape of regular polygons, the only possible choice of regular polygons are equilateral triangles, squares, and hexagons. In
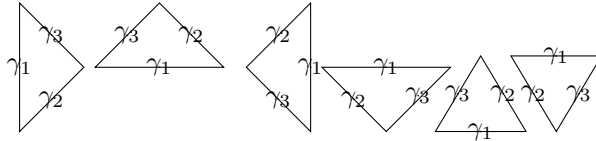
Figure 1: Four right triangular tiles and two equilateral triangular tiles

this paper, we discuss the self-assembly of triangles and other shapes by triangular tiles, more specifically, of shape of equilateral triangles and of right triangles, respectively.

In Section 2, we will introduce the definition of triangular tile assembly system. In Section 3, we discuss the computational power of the triangular tile assembly system, and show that it is Turing universal. In Section 4, we compare the square tile assembly system and triangular tile assembly system in the aspect of shape complexity and show that the two types of system are not comparable. In Section 5, we discuss the assembly of triangles. In the last section, we summarize the obtained results.

2 Definitions

All discussion in this paper is on a two-dimensional plane. Before we discuss the right triangular tiles and the equilateral triangular tiles respectively, we first give a uniform definition of the *Tile Assembly Model* (TAM).

Similar to the square tiles, we define a *triangular tile* to be an triangle of particular shape (right triangle or equilateral triangle) with each side being colored from the set Σ of "glues". Without loss of generality, we assume that the shortest side of a triangular tile is of unit length. We also assume that a triangular tile cannot be rotated nor flipped over. Both square tiles and triangular tiles are called tiles.

The particular non-interactive glue is denoted by ϕ and we always assume $\phi \in \Sigma$. The *temperature* τ is a real number, which presents under which the assembly is proceeded, and the set of all valid temperature is denoted by $\mathcal{W}$. A *strength function* $g : \Sigma \times \Sigma \to \mathcal{W}$ is defined such that $g(\gamma, \gamma') = g(\gamma', \gamma)$ and $g(\phi, \gamma) = 0$. In particular, we are interested in the discrete case $\tau \in \mathcal{N}$, $\Sigma = \Gamma \times \mathcal{N}$ and $g((a, n), (a', n')) = n$ if $a = a', n = n'$ otherwise $g((a, n), (a', n')) = 0$, where $\mathcal{N} = 0, 1, \dots$ are non-negative integers.

We say two tiles can *stick* together if they can be physically put adjacent by the sides γ and γ' of the same length such that $g(\gamma, \gamma') \geq \tau$. A tile can stick to a set of tiles if they can be physically put adjacent by the sides γ_i and γ'_i of the same length such that $\sum_i g(\gamma_i, \gamma'_i) \geq \tau$. A *super-tile* is a set of tiles that stick to each other such that no two tiles overlap and for any two tiles there is a path of sticked edges between them. We also call a single tile super-tile.

A *tile assembly system* is a tuple $S = (T, s, g, \tau)$, where T is a finite set of tiles, $s \in T$ is a particular super-tile called *seed*, g is a strength function, and τ is the temperature. The produce of a tile assembly system is a super-tile st such that there is a super-tile sequence $s = st_0, st_1, st_2, \dots, st = st_n$, where st_{i+1} is obtained by stick one tile in T to st_i under temperature τ and no tile in T can be stick to st_n to obtain a bigger super-tile. A tile assembly system is *deterministic* if its produce is unique regardless of the different choice of super-tile at each step.

In analogy to the self-assembly of full square, in what follows we will study the self-assembly of full triangles and other full shapes by right triangular tiles and equilateral triangular tiles, respectively. Here "full" means the pair of common edges of every two adjacent tiles in the produce has a positive strength.

Right triangular tiles are triangular tiles of the shape of right triangles with the right angle point to four possible directions as illustrated in Figure 1. More formally, a right triangular tile is described by $(\gamma_1, \gamma_2, \gamma_3, k)$, where $\gamma_i \in \Sigma$ are glues on sides of the tile in the counter-clockwise order starting from the longest side and $k \in \{ \mathtt{e}, \mathtt{n}, \mathtt{w}, \mathtt{s} \}$ presents the direction of the right angle. *Equilateral triangular tiles* are triangular tiles of the shape of equilateral triangles that are either in an upward position or in a downward position as illustrated in Figure 1. More formally, it is described by $(\gamma_1, \gamma_2, \gamma_3, k)$, where $k \in \{ \mathtt{u}, \mathtt{d} \}$ presents the two positions and $\gamma_i \in \Sigma$ are glues on sides of the tile in the counter-clockwise order starting from the horizontal side.
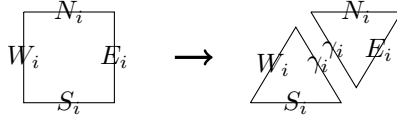
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Figure 2: Conversion of a Wang system with square tiles into an equivalenct Wang system with equilateral triangular tiles

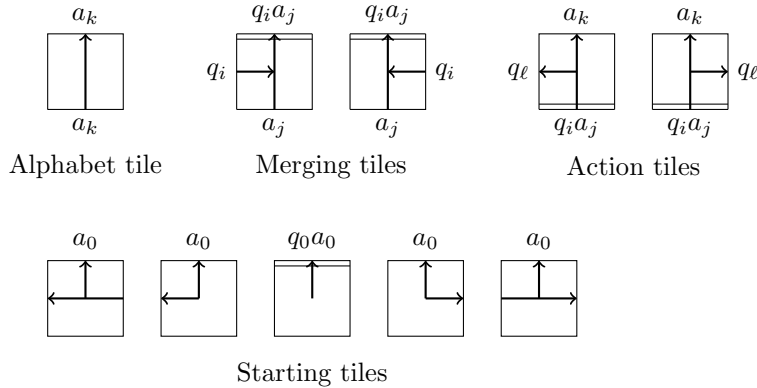Alphabet tile   Merging tiles   Action tiles

Starting tiles

Figure 3: Tiles of a square TAM which simulates a given Turing Machine at temperature $\tau = 2$. A tile can stick to its neighbor via a single-lined edge with a glue strength 1 or via double-lined one with strength 2, but the glue works only when the abutting edges share the same label and the directions of their arrow heads (if any) must match (head with tail).

3 Computational Complexity

Tiling a plane is equivalent to attaching tiles onto a lattice of a coordinate system on the plane. The choice of coordinate system is arbitrary, but square tile systems are to choose the rectangular coordinate system C_R. In contrast, the oblique coordinate system is rather natural as a pasted board of triangular tiles. It seems reasonable to say that the oblique coordinate system $C_{\pi/3}$ whose two axes intersect with $\pi/3$ is the best choice for equilateral triangular tiles. The right triangular tile accords with both rectangular and oblique coordinate systems because it can tessellate unlike equilateral triangular tiles. The conversion among these coordinate systems can be done by affine transformations.

As implied in Figure 2, so far as Wang tile system is concerned, whether the tile shape is square, equilateral triangle, or right triangle does not matter because the Wang tile system does not have the notion of growth by time or temperature, and imposes that any abutting edges have to have the same glue. Several problems on the computational complexity of Wang tile system was studied by Robinson in 1971 [4]. Among them, one important problem is the tiling full plane problem: given a Wang tile system, decide whether any product of that system is not a full plane. The argument so far should make it clear that we can obtain analogous results for Wang system with triangular tiles. For example, tiling full plane problem is undecidable for a given Wang system with (equilateral, right) triangular tiles.

This conversion may still work for non-deterministic tile assembly models, but it does not work any more for deterministic ones. Let us verify this statement by trying to simulate a Turing machine by the triangular TAM thus obtained. Based on the conversion, the tile in Figure 3 which merges the state q_i from the right to the letter a_j is split into $(a_j, \gamma, \phi, \mathtt{u})$ and $(q_i a_j, \gamma, q_i, \mathtt{d})$. What is important is that two inputs of the square merging tile a_j, q_i are now separated onto the two triangular tiles, and cannot cooperate until one of the tiles is stuck to the super-tile.
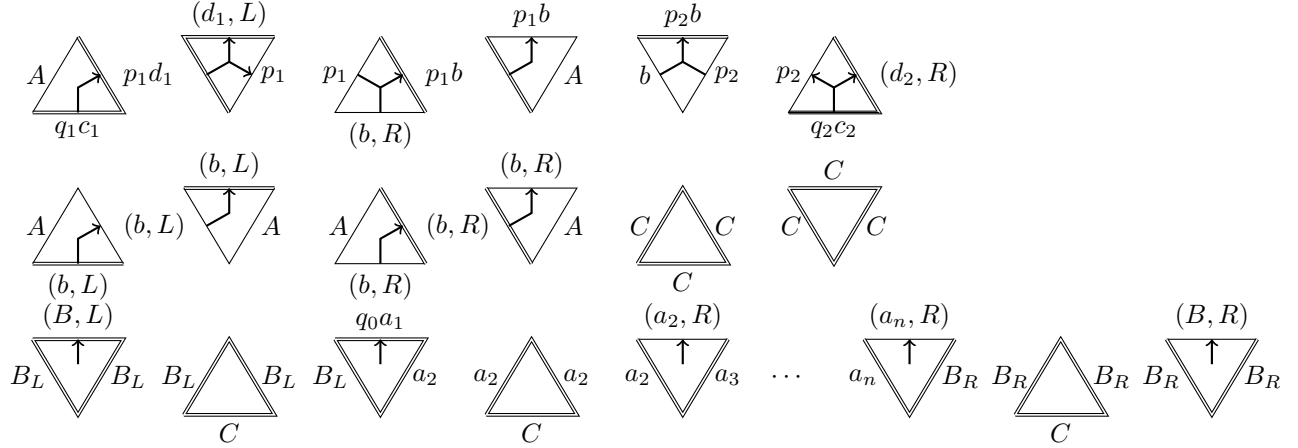
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Figure 4: Equilateral triangular tiles for the simulation of Turing machines, where $a_1 \cdots a_n$ is the input, $b \in \Sigma$, $\delta(q_1, c_1) = (p_1, d_1, \mathtt{L})$, and $\delta(q_2, c_2) = (p_2, d_2, \mathtt{R})$. (top) action tiles and merging tiles; (middle) alphabet tiles for letters to the left of TM head (indicated by L) and for letters to the right (R). The two tiles with the label C are used to fill the third and fourth quadrants; (bottom) tiles for initialization with the third tile as seed;

This failure means that the conversion requires some modification for the deterministic triangular TAM construction. In the following, we will prove that the triangular tile assembly system is Turning universal in the sense that the tiling full plane problem can simulate the halting problem. Throughout these proofs, it will be elucidated what modification to be required.

Theorem 1. *The deterministic equilateral triangular tile assembly system is Turing universal at temperature* $\tau \geq 2$.

Proof. We simulate a given deterministic Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, \mathtt{B}, F)$ by a deterministic equilateral triangular TAM whose tile set is shown in Figure 4. Without loss of generality, we can assume that M always moves its head when it transits.
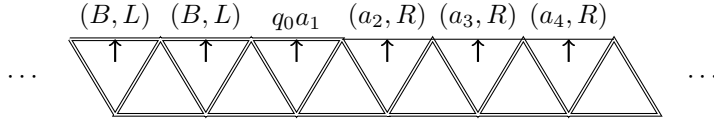

Using the tiles on the bottom row of Figure 4, The initial configuration $\cdots \mathtt{B} q_0 a_1 a_2 \cdots a_n \mathtt{B} \cdots$ self-assembles from the seed $(q_0 a_1, B_L, a_2, \mathtt{d})$ in a straightforward manner as shown just above. Each letter is coupled either with the indicator L if the letter is to the left of the head or with R otherwise. Note that the top edges with the TM head or to the left of the head are double-lined, and hence are bound to their matching bottom edges with strength 2. Thus, for instance, the upward alphabet tile with L at its bottom can stick to these top edges without any cooperation so long as their letters match. This is not the case for the edges to the right of the head because their glue strength is 1.

Let us consider the transition $\delta(q_0, a_1) = (q_1, b_1, \mathtt{R})$ first. Via the edges with strength 2, the upward alphabet tiles simultaneously stick to the edges located to the left of the TM head. In order for them to extend further by using the corresponding downward alphabet tiles, an action tile $((q_0 a_1, q_1 b_1, A, \mathtt{u})$ in this case) has to be stuck to the super-tile. The action tile changes the state q_0 and the letter a_1 according to the transition to q_1 and b_1 deterministically, and its corresponding downward tile branches the letter up and the state to the right. Now the merging tile $((a_2, R), q_1 a_2, q_1, \mathtt{u})$ can attach by the cooperation of left and bottom edges, and the attachment of its corresponding upward tile immediately follows. The letters to the right of TM head are extended one by one in this manner.

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(B,L) (B,L) (b_1,L) q_1a_2 (a_3,R)

q_1

$\cdots$ q_0a_1 (a_2,R) (a_3,R) (a_4,R) $\cdots$
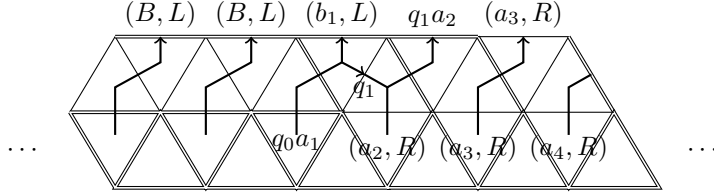
The transition $\delta(q_1,a_2) = (q_2,b_2,\mathtt{L})$ is simulated essentially in the same manner as the previous simulation so that it may suffice to illustrate it as follows:

(B,L) (B,L) q_3b_1 (b_2,R) (a_3,R)

q_3 b_2

(b_1,L) q_1a_2

q_1

$\cdots$ q_0a_1 (a_2,R) (a_3,R) (a_4,R) $\cdots$
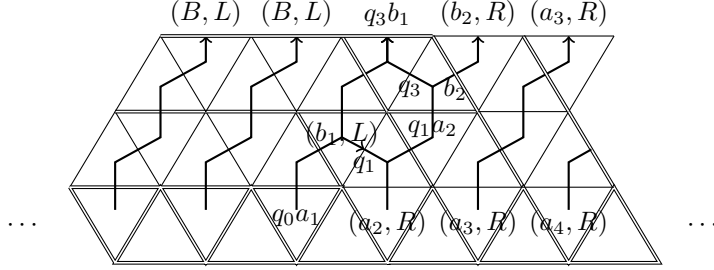
This Turing machine simulator consists of at most $2n+3+4|\Sigma|+m(1+3|\Sigma|)$ tiles, where n is the length of the input $a_1\cdots a_n$, and m is the number of transitions defined in this Turing machine M. □

This simulation keeps tiling the plane upward until it reaches some halting configuration, i.e., the head is in a state q and is on the cell with a letter b such that $\delta(q,b)$ is not defined. So the undecidability of halting problem of Turing machine leads us from this theorem to the following corollary.

Corollary 2. *It is undecidable whether a given deterministic equilateral triangular tile assembly system produces a super-tile other than full plane.*

For any equilateral triangular tile assembly system $S = (T,s,g,\tau)$, we define the *"flattened"* right triangular assembly system $\mathcal{F}(S) = (T',f(s),g,\tau)$, where $T' = \{\,f(t),t \in T\,\}$, $f(\gamma_1,\gamma_2,\gamma_3,\mathtt{u}) = (\gamma_1,\gamma_2,\gamma_3,\mathtt{n})$, and $f(\gamma_1,\gamma_2,\gamma_3,\mathtt{d}) = (\gamma_1,\gamma_2,\gamma_3,\mathtt{s})$. Then S produces a super-tile that is not the full plane if and only if $\mathcal{F}(S)$ also produces a super-tile that is not the full plane. Then the following corollaries of Theorem 1 hold.

Corollary 3. *Deterministic right triangular tile assembly system is Turing universal at temperature τ for $\tau \geq 2$.*

Corollary 4. *It is undecidable whether a given deterministic right triangular tile assembly system produces a super-tile other than full plane.*

One advantage of right triangles over equilateral ones is that right triangles can tile the square grid, and actually there are two ways to fill a square being rotated by $\pi/4$ with right triangles: east and west triangles or north and south triangles. This fact enables us to handle more intuitively the "input-split" problem which the equilateral triangular TAM has already encountered. That is, the square tile which merges a state from the left is split into half from its left-top to right-bottom, while the tile which merges from the right is cut from right-top to left-bottom. Figure 5 illustrates the right triangular TAM designed according to this idea, which simulates a deterministic Turing machine $M = (Q,\Sigma,\Gamma,\delta,q_0,\mathtt{B},F)$ on an input $a_1a_2a_3\ldots a_n$. One can verify by definition that the given system is a deterministic right triangular tile assembly system with at most $2n+4+4|\Sigma|+m(1+2|\Sigma|)$ tiles (slightly better than the flattened right triangular TAM), where n is the length of the input of the Turing machine M and m is the number of transitions defined in the Turing machine M.

The simulations of Turing machines by TAMs with different shapes given in this section negates the idea that the right triangular TAM or even equilateral triangular TAM might be completely equivalent to the square TAM, topologically, in spite of the different tile shapes. We will proceed this investigation further in the next section.
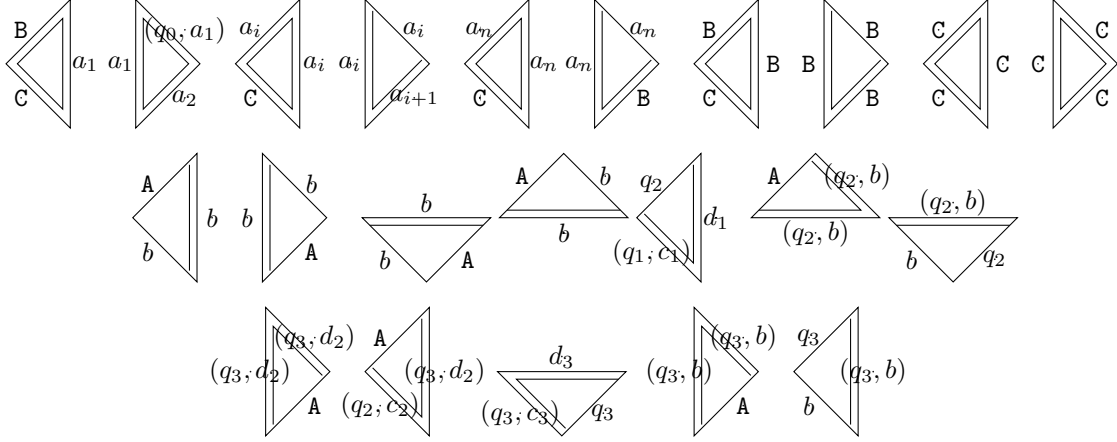
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Figure 5: Right triangular tiles for the simulation of Turing machines with the seed $(a_1, \mathtt{B}, \mathtt{C}, \mathtt{w})$. In this figure, a_i are the input, $b \in \Sigma$ iterate all letters, $q_i \in Q \setminus F$, $\delta(q_1, c_1) = (p_1, d_1, \mathtt{L})$, $\delta(q_2, c_2) = (p_2, d_2, \mathtt{N})$, and $\delta(q_3, c_3) = (p_3, d_3, \mathtt{R})$.

4   Shape Complexity

We can a shape *compatible* with a given type of tile assembly system, if the region occupied by that shape on the two dimensional plane can be tiled geometrically by the tiles. It is obviously that if a shape is not compatible, then no super-tile of that shape can be produced by tile assembly systems. For example, a pie cannot be assembled by unit square tiles nor triangular tiles. So in the remaining discussion, we only consider the assembly of compatible shapes.

Proposition 5. *Any compatible shape can be produced by a non-deterministic triangular tile assembly system with $O(1)$ tiles, and by a deterministic triangular tile assembly system with A tiles, where A is the totally number of tiles to geometrically assemble that shape.*

Proof. First we consider the equilateral triangles in the non-deterministic cases. Consider the set of tiles $T = \{ (a, b, c, k) : a, b, c \in \{ \phi, g \}, k \in \{ \mathtt{u}, \mathtt{d} \} \}$. All glue of g are of strength 1 and temperature is $\tau = 1$. The seed and assembly process is as follows: the super-tile of given shape is assembled according to the geometrical division of the region. This can be done since the shape is compatible. At each step, a tile sticks to the super-tile in such a way that if the tile is surrounded by other tiles in the completed region, then every edge of that tile is of glue g; otherwise, the edge that composes the border of that region is of empty glue ϕ.

If we make each pair of stick sides with unique glue, then the shape can be assembled deterministically by $O(A)$ tiles.

The right triangles case is similar.   □

Proposition 5 can be generalized to tiles of other shapes, such as square tiles. In what follows, we only consider deterministic tile assembly systems.

A right triangular tile assembly system T is called a *division* of a square tile assembly system S if for any tile s in S, there is a pair of tiles t, t' such that on temperature $\tau \geq 1$ tiles t, t' can produce s with $\pi/4$ rotation; and for any tile t in T, there is a tile t' in T and a tile s in S such that on temperature $\tau \geq 1$ tiles t, t' can produce s with $\pi/4$ rotation. By the definition, division of a square tile assembly system may not be unique, and a right triangular tile assembly system can be division of two different square tile assembly system. The number of tiles in the two systems satisfies the inequality

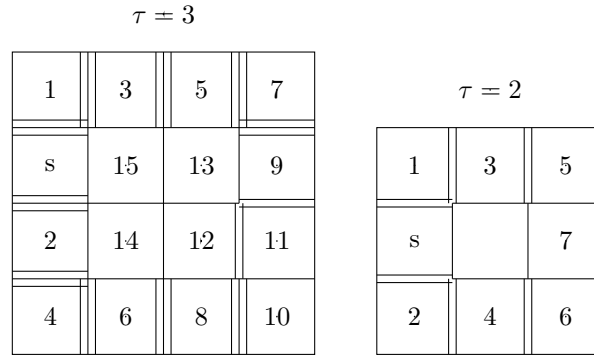$$2\sqrt{\#S} \leq \#T \leq 4 \cdot \#S,$$

Figure 6: Two counter-examples that square tile assembly system cannot be simulated by triangular tile assembly system. Each glue is unique and thus the label is omitted.

where # presents the number of tiles in each system. A equilateral triangular tile assembly system T is called a division of a square tile assembly system S if the flattened tile assembly system $\mathcal{F}(T)$ is a division of S.

Lemma 6. *There exists a square tile assembly system S such that no division of S can produce the same shape with $\pi/4$ rotation.*

Proof. We presents two examples here. The produce of the two square tile assembly systems are illustrated in Figure 6, where each glue is unique and the strength is illustrated by the number of parallel edges. The temperatures are of $\tau = 3$ and $\tau = 2$ respectively. The number of divisions of the square tile assembly system is finite. One can verify that none of them produce the same shape with the original system. □

The right super-tile in Figure 6 is of shape with a missing tile in the middle, and we call it has *"hole"*. More formally, we say a super-tile has no hole if it is full and for every closed path of tiles, all enclosed region is occupied by tiles.

Lemma 7. *For any square tile assembly system S under temperature $\tau = 1$ or under temperature $\tau = 2$ with no hole in the produce, there is a division of S that can produce the same shape with $\pi/4$ rotation.*

Proof. For $\tau = 1$, the proof is straightforward. For any square tile s_i with glues $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ (on east, north, west, south sides, respectively) in S, we replace it with a pair of right triangular tiles $(i, \gamma_1, \gamma_2, \mathtt{n})$ and $(i, \gamma_3, \gamma_4, \mathtt{s})$. Then the new right triangular tiles is a division of S and produce the same shape with $\pi/4$ rotation.

For $\tau = 2$, now we assume there is no hole in the produce of S.

First we prove that there is an assembly process $st_0, st_1, st_2, \ldots, st_n$ such that every super-tile st_i in the process has no hole in it. Otherwise, we pick such a process that the steps of the first appearance of a hole super-tile is the largest among all assembly process. Let st_i be the first appearance of a hole super-tile. Then there is a tile t in the hole region that will stick to the super-tile later and there are two adjacent edges that can stick to that tile due to the fullness of the produce. So, we can add t to the super-tile st_{i-1} and get a new st_i' that does not have hole, which contradiction to the choice of the process.

Now we prove that for the assembly process without hole super-tile, the new tile can stick to the super-tile at each step by two adjacent edges. Otherwise, suppose st_{i-1} becomes st_i by sticking t and t only stick to st_{i-1} either by north and south sides or by east and west sides. Without loss of generality, suppose it is by north and south sides. Since the produce is full, there is no tiles on the east and on the west sides, or t can stick by two adjacent edges. But in this case, since st_{i-1} is connect, st_i must contains a hole, which contradicts the fact st_i has no hole.

Since new tile can stick to the super-tile at each step by two adjacent edges, we can add a pair of right triangular tiles to simulate that square tiles; and we let the strength on the cutting edges be $\geq \tau$. Do so for the whole assembly process, and we get a new right triangular tiles, which is a division of S and produce the same shape with $\pi/4$ rotation. □
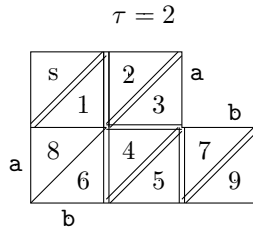
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$$\tau = 2$$

Figure 7: A counter-examples that triangular tile assembly system cannot be simulated by square tile assembly system. Each glue, unless mentioned, is unique and thus the label is omitted. For convenience, the picture is rotated by $\pi/4$.

By the previous two lemmas, we see that square tile assembly systems can be simulated by their division only under certain conditions. Not we discuss the relation between the two types of tile assembly systems in another direction: whether every right triangular tile assembly system can be simulated by a square triangular tile assembly system, where we assume the produce of the system is compatible with the tiles.

Lemma 8. *There exists a right triangular tile assembly system T such that there is no square tile assembly system S that produce the same shape with $\pi/4$ rotation.*

Proof. An example is illustrated in Figure 7, where the strength is illustrated by the number of parallel edges. The system is under temperature $\tau = 2$ and the top-left tile is the seed. Let S be any square tile assembly system that generate a super-tile as in the example. Then S will continue growing by sticking a tile on the left bottom corner to the right top corner. Therefore, the super-tile in the example cannot be produced by square tile assembly system. □

Lemma 9. *For any right triangular tile assembly system T under temperature $\tau = 1$, there is a square tile assembly system S that can produce the same shape with $\pi/4$ rotation.*

Proof. For $\tau = 1$, we construct a square tile assembly system S from T as follows: for every pair of right triangular tiles $(\gamma_1, \gamma_2, \gamma_3, \mathtt{n}), (\gamma_1, \gamma_4, \gamma_5, \mathtt{s})$ or $(\gamma_1, \gamma_2, \gamma_3, \mathtt{e}), (\gamma_1, \gamma_4, \gamma_5, \mathtt{w})$ in T, where $\gamma_1 \neq \phi$, we add a new square tile with glues $\gamma_4, \gamma_5, \gamma_2, \gamma_3$ or $\gamma_5, \gamma_2, \gamma_3, \gamma_4$ (on east, north, west, south sides, respectively) to S. Then the new square tile assembly system produce the same shape with $\pi/4$ rotation. □

To compare the produces of two tile assembly system, we not only compare the shape of the produce, but also compare the glues on shared common edges, including both those on border and those inside the super-tiles, with possible affine transformation on the shape, which includes rotation, scaling, shift, and their compositions. We call the power of produce certain super-tiles the *shape complexity* and say one system has greater power than another system if every system produce of the former type with compatible shape is the produce of some system of the latter type.

For every equilateral triangular tile assembly system T, there is a right triangular tile assembly system $\mathcal{F}(T)$ such that the produces of two system is equivalent up to an affine transformation. There are two more types of tiles in right triangular tile assembly system, which cannot be simulated by equilateral triangular tiles. So we can say the shape complexity of equilateral triangular tile assembly system is less than that of right triangular tile assembly system.

The example given in Lemma 8 is a flattened equilateral triangular tile assembly system. In other words, there exists equilateral triangular tile assembly system which cannot be produced by square tile assembly system even under affine transformations. By Lemma 6 and Lemma 8, we have the follow theorem.

Theorem 10. *The square tile assembly systems and the triangular tile assembly systems are not comparable in the sense of shape complexity.*

5   Assembly of Triangles

Without loss of generality, we consider the assembly of an upright full triangle. For downward full triangle, one can simply define a new triangular tile system by flip-over each tiles in the original triangular tile system.
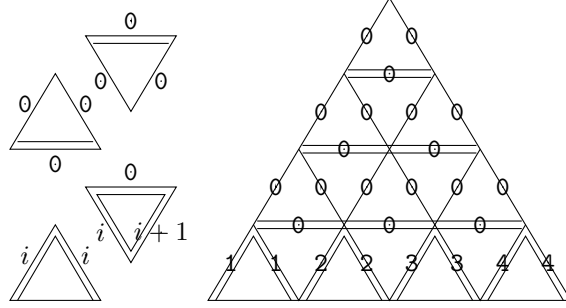
Figure 8: A triangular tile assembly system of $2N - 1$ tiles produces a full triangle, where $1 \leq i \leq N$. On the right is an example for $N = 4$

Proposition 11. *For temperature $\tau = 1$ the minimal number of tiles to assemble a full triangle with shortest edge of length N is N^2, including $N(N+1)/2$ upright triangular tiles and $N(N-1)/2$ downright triangular tiles.*

Proof. Without loss of generality, we consider the assembly of equilateral triangles by equilateral triangular tiles. For the case of right triangles, we can treat it as a flattened equilateral triangles.

By Proposition 5, there exists a system of N^2 tiles to assemble the required full triangle. To show that it is optimal, suppose there is a system $(T, s, g, 1)$ with less tiles. Then by the pigeon hole principle, there are two tiles t_1, t_2 in the produce that are the same. Since the produce is a full triangle, there is a non-crossing path of tiles from s to t_1 and from t_1 to t_2, respectively. Since the temperature $\tau = 1$, there is a possible assembly process that starts from s and sticks each tile along the path from s to t_1 and then from t_1 to t_2. After t_2 sticks to the super-tile, again sticks each tile along the path from t_1 to t_2 (t_1 itself is not included). So the produce of the system is a infinite structure. Since the system is deterministic, the produce cannot be a triangle, which contradicts the assumption. So the system with N^2 tiles is optimal. ◻

Now we consider the temperature $\tau \geq 2$. First we show how to use $2N - 1$ triangular tiles, including $N + 1$ upright tiles and N downright tiles assemble a full triangular.

Proposition 12. *For temperature $\tau = 2$ there is a triangular tile assembly system of $2N - 1$ tiles that produces a full triangular with shortest edge of length N.*

Proof. The system is illustrated in Figure 8. The construction here works for both equilateral triangular tiles and right triangular tiles. ◻

Using the same technique of square tile assembly for $N \times N$ squares [5], the following result follows.

Proposition 13. *There is a right triangular tile assembly system of $O(\log N)$ tiles that produces a full right triangular with shortest edge of length N.*

Proof. The idea is that using a seed row of length $n = \lceil \log N \rceil$ to construct a $(n-1) \times (N-n+1)$ rectangle super-tile by counting from $(2^n - N + n + 2)/2$ to 2^{n-1} with duplicate copies. Then the rectangle is completed by filling tiles to make a right triangle. The temperature is $\tau = 2$ and tiles is illustrated in Figure 9, where $(s1, \phi, \mathtt{l}, \mathtt{w})$ is the seed. There are in total $2n + 37$ tiles. ◻

Using the same technique of base conversion as appeared in the square tile assembly [1] to count the integer represented by tiles, the bound on the minimal number of tiles required can be improved to $O(\log N / \log \log N)$, which is optimal; the construction is under temperature $\tau = 3$.

Corollary 14. *There is a right triangular tile assembly system of $O(\log N / \log \log N)$ tiles that produces a full right triangular with shortest edge of length N.*
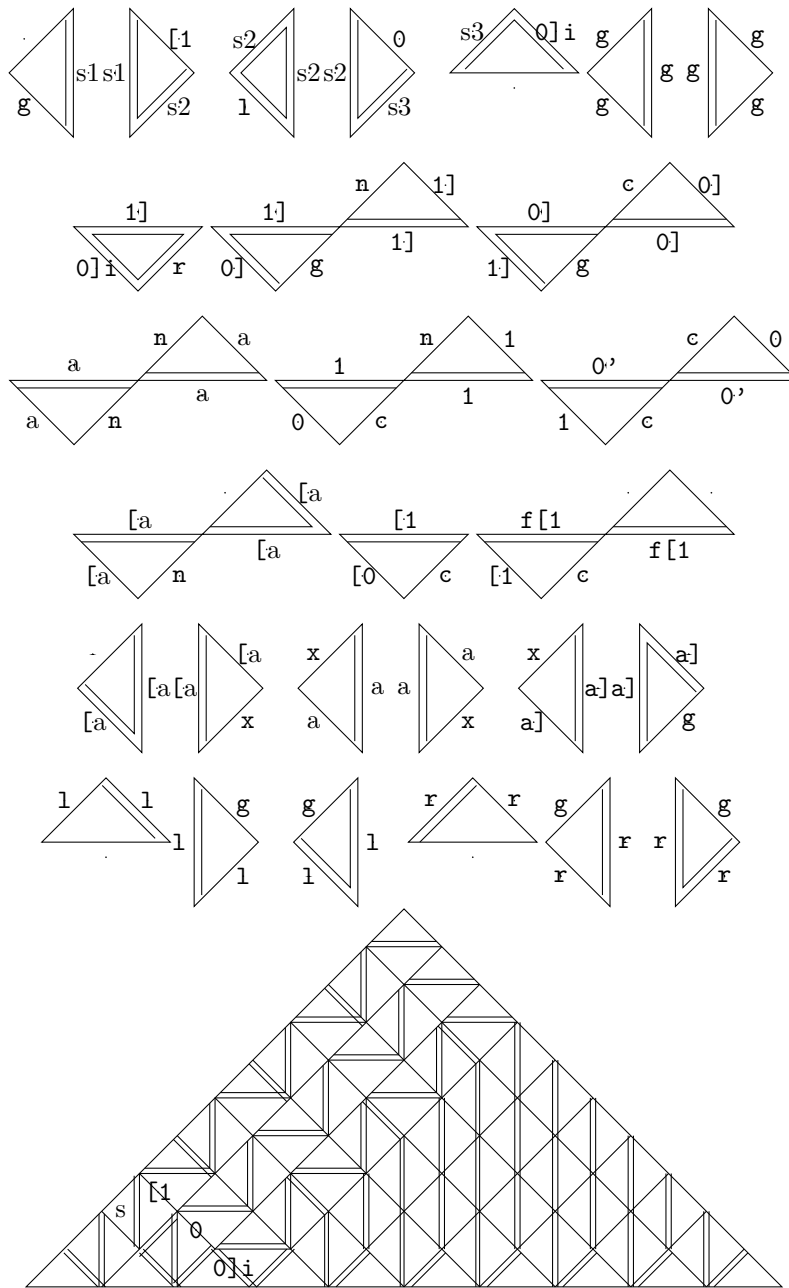
Figure 9: The $O(\log N)$ tiles, $a \in \{0, 1\}$, and the produce for $N = 10, n = 4$. For simplicity, the label on the super-tile is omitted.

6   Conclusion

Square tile assembly system is discussed in the literatures widely. In this paper, we studied the triangular tile assembly system. We showed that the triangular tile assembly system is also Turning universal. The halting problem can be reduced to the tiling full plane problem.

Compared to the square tile system, the triangular tile system need more tiles to assemble a large compatible structure due to the fact that a triangular tile has less edges than a square tile. In general, as we showed, the two type of assembly system is not comparable in the shape complexity. More precisely, there exists a square tile assembly system S such that no division of S produces the same shape with $\pi/4$ rotation; and there exists a triangular tile assembly system T such that no square tile assembly system produces the same shape, which is compatible with square tiles, with same border glues with $\pi/4$ rotation.

We also discussed the assembly of triangles and the number of tiles required to assemble a triangle with minimal edge of length N is $O(\log N/\log\log N)$, which is of the same order as those of square tiles. The techniques used here is from that of assembly of squares.

The model we used in this paper is of fixed temperature, unit growing (at each step, only a single tile stick to the super-tile), and irreversible (once tiles stick together, they will not break in the further). There are other possible choice of models. For example, if we allow variable temperature and reversible process as discuss on square tiles [1], then in exactly the same way to the assembly of squares, one can prove without difficulty that $O(1)$ tiles is enough to assemble arbitrary large compatible triangles; in that case the time sequence is of length $O(\log N)$.

All the result presented in the paper is based on theoretical study. It will be interesting to assemble a triangle structure using triangular tiles in the laboratory.

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